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The Differential Equations of Ballistics

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VIII. *The Differential Equations of Ballistics.*

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(*Communicated by Sir* George Hadcock, *F.R.S.*)

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1. *Introduction.*

The differential equations arising in most branches of applied mathematics are linear equations of the second order. Internal ballistics, which is the dynamics of the motion of the shot in a gun, requires, except with the simplest assumptions, the discussion of non-linear differential equations of the first and second orders.

The writer has shown in a previous paper* how such non-linear equations arise when the pressure-index α in the rate-of-burning equation differs from unity, although only the simplified case of non-resisted motion was there considered. It is proposed in the present investigation to examine some cases of resisted motion taking the pressure-index equal to unity, to give some extensions of the previous work, and to consider, so far as is possible, the nature and the solution of the types of differential equations which arise. A discussion of the resisted motion involving a general rate-of-burning law would appear to be too complicated for any attempted treatment to be profitable, so that the investigation is divided into two parts: (a) resisted motion with a linear rate-of-burning law $\alpha = 1$; (b) non-resisted motion with $\alpha \neq 1$.

The internal ballistic problem leads to *linear* differential equations only in the simplified case where direct proportionality between the rate-of-burning of the propellant and the pressure is assumed, and consideration of all perturbations such as band resistance and friction is excluded. In all other cases non-linear equations are met with, and, since, for example, the problem of resisted motion will surely be conceded to be of importance to the practical ballistician, it is thought that some account of the theory and the attempts at its elucidation will not be without value.

Differential equations of a type somewhat similar to those discussed in this paper arise in astrophysical investigations; thus Emden's general polytropic equation for the equilibrium of a gravitating gas sphere is†

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y^n = 0,$$

or, with $x = 1/\xi$,

$$\frac{d^2y}{d\xi^2} + \xi^{-4}y^n = 0.$$

* 'Phil. Trans.,' A, vol. 227, p. 345 (1928).

† *Vide* Fowler, 'Mon. Not. R. Astr. Soc.,' vol. 91, p. 63 (1931).

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If $n = -1$ this agrees in type with one of the equations of Internal Ballistics, although this value of n, of course, precludes the equation from having any direct astrophysical meaning.

It is as well to state here that, so far as is possible, the symbols and notation employed will be the same as in the writer's previous paper (*loc. cit.*, p. 1) which will be denoted throughout by I.B. Frequent reference will be made to it, and the present paper is to be regarded largely as a sequel.

2. *The Resisted Motion in the Gun.*

When a shot moves along the bore of a gun under the pressure of the propellant gases there is considerable resistance to the motion arising principally from the friction between the driving band and the bore, as well as, to begin with, the resistance offered by the band during engraving.

In the idealised ballistic problem such resistance is neglected, but it is of obvious practical importance to consider its effect. It is essential for the purpose of simplifying the analysis to devise a method of continuous integration of the equations of motion of the projectile, so that no distinction is made in form between the resistance to motion during band engraving and that arising afterwards.

The linear rate-of-burning law of propellant is assumed, so that $\alpha = 1$, and the resistance to motion is denoted by R.

We use the equations of Scheme I: (11) to (13) of I.B., p. 352, which, introducing R and putting $\alpha = 1$, become

$$p = \lambda \phi(f)/x$$

$$\mathrm{D}\, df/dt = -\beta p$$

$$\mu\, d^2x/dt^2 = \mu\, dv/dt = p - \mathrm{R},$$

where v is the shot velocity and R is regarded, at present, as an unknown function of f, since in the above equations f is taken as the independent variable. (For the definition of f see I.B., p. 350.)

From these equations, eliminating t between the last two we have

$$\mathrm{R}/p = 1 + \frac{\mu\beta}{\mathrm{D}}\frac{dv}{df}, \qquad (1)$$

and differentiating the first with respect to f it may be written as

$$\frac{dx}{df} = \lambda \frac{d}{df}\left[\frac{\phi(f)}{p}\right] = -\frac{\mathrm{D}v}{p\beta}. \qquad (2)$$

Eliminating p between (1) and (2) gives

$$\frac{\lambda\beta}{\mathrm{D}}\frac{d}{df}\left[\frac{\phi(f)}{\mathrm{R}}\left(1 + \frac{\mu\beta}{\mathrm{R}}\frac{dv}{df}\right)\right] = -\frac{v}{\mathrm{R}}\left(1 + \frac{\mu\beta}{\mathrm{D}}\frac{dv}{df}\right),$$

which becomes

$$R\phi(f)\frac{d^2v}{df^2}+\left[R\phi'(f)-R'\phi(f)+\frac{D}{\lambda\beta}Rv\right]\frac{dv}{df}$$
$$+\frac{D^2}{\lambda\mu\beta^2}Rv+\frac{D}{\mu\beta}[R\phi'(f)-R'\phi(f)]=0, \quad . \quad (3)$$

primes denoting differentiation with respect to f. We now proceed to investigate a form for R which will permit of the integration of this equation.

If the equation has a first integral, it must clearly be of the form

$$\frac{dv}{df}=L+Mv+Nv^2, \quad \ldots \ldots \quad (4)$$

where L, M, N are functions of f.

Differentiating (4) and replacing v^2 where it occurs by

$$\left(\frac{dv}{df}-L-Mv\right)\Big/N,$$

we find

$$\frac{d^2v}{df^2}=(L'-LN'/N)+(M'-MN'/N)\,v+(M+2Nv+N'/N)\frac{dv}{df}. \quad . \quad . \quad (5)$$

Comparison with (3) gives, writing $\phi(f)$, $\phi'(f)$ as ϕ, ϕ' for brevity

$$\left.\begin{aligned} L'-\frac{LN'}{N}&=\frac{D}{\mu\beta}\left(\frac{R'}{R}-\frac{\phi'}{\phi}\right)\\ M'-\frac{MN'}{N}&=-\frac{D^2}{\lambda\mu\beta^2}\frac{1}{\phi}\\ M+\frac{N'}{N}&=\frac{R'}{R}-\frac{\phi'}{\phi}\\ 2N&=-\frac{D}{\lambda\beta}\frac{1}{\phi}\end{aligned}\right\}, \quad \ldots \ldots \quad (6)$$

four equations with four unknowns L, M, N, R.

The last equation gives $N'/N=-\phi'/\phi$, so that from the third, $M=R'/R$, and from the second $(d/df)(M\phi)=-D^2/\lambda\mu\beta^2=-c$, say, so that $M\phi=A-cf$, where A is an arbitrary constant, and therefore $R'/R=(A-cf)/\phi$.

Finally, the first equation of (6) gives

$$\frac{d}{df}(L\phi)=\frac{D}{\mu\beta}(A-cf-\phi'),$$

i.e.,

$$L\phi=B+\frac{D}{\mu\beta}(Af-\tfrac{1}{2}cf^2-\phi),$$

where B is another arbitrary constant.

Thus L, M, N, are determined and R can be found when $\phi(f)$ is given.

The motion begins when the gas pressure p is equal to the resistance R, so that initially f has a value f_0 which is less than unity, since some of the propellant must be converted into gas before motion starts. When the propellant is all converted into gas $f = 0$.

It is natural to expect the resistance to diminish as the motion proceeds so that R should decrease as f decreases, or R′ ($= d\text{R}/df$) should be positive, *i.e.*, we must have $\text{A} > cf_0$ since f decreases from f_0 to zero during the motion.

Taking $\phi(f) = (1 - f)(1 + \theta f)$ (*vide* I.B., p. 350), we find

$$\left.\begin{aligned} \text{R} &= \text{C}\,\frac{(1+\theta f)^{\frac{\text{A}\theta + c}{\theta(1+\theta)}}}{(1-f)^{\frac{\text{A}-c}{1+\theta}}} \\ \text{with}\quad \text{R} &= \frac{\text{C}e^{cf}}{(1-f)^{\text{A}-c}} \quad \text{for } \theta = 0 \\ \text{and}\quad c &= \frac{\text{D}^2}{\lambda\mu\beta^2} \end{aligned}\right\}, \qquad (7)$$

A, C, being arbitrary constants.

When the propellant is completely burnt, $f = 0$, and so $\text{R} = \text{C}$ for any further travel.

There are apparently three constants A, B, C left arbitrary, but the constant C can be determined in terms of A by consideration of the fact that if p_0 is the shot-start pressure and f_0 the corresponding value of f, then $\text{R} = p_0$ when $f = f_0$.

Also from the first of the equations of Scheme I quoted above, $p_0 = \lambda\phi(f_0)/l$ since l is the initial value of x (for definition of l, see I.B., p. 351), so that

$$\lambda\phi(f_0)/l = \text{C}(1+\theta f_0)^{\frac{\text{A}\theta+c}{\theta(1+\theta)}}\Big/(1-f_0)^{\frac{\text{A}-c}{1+\theta}}.$$

The constant B is also determinable in terms of A, as will appear shortly.

Equation (4) now becomes

$$\frac{dv}{df} = -\frac{\text{D}}{\mu\beta} + \frac{\text{B}}{\phi} + \frac{\text{D}}{\mu\beta}\left(\frac{\text{A}f - \frac{1}{2}cf^2}{\phi}\right) + \frac{\text{A} - cf}{\phi}\,v - \frac{\text{D}}{2\lambda\beta}\,\frac{v^2}{\phi}, \qquad (8)$$

which is of the Riccati type, so that, if any particular solution is known, the complete primitive can be obtained by quadratures.

The initial conditions are $f = f_0$, $v = 0$, and $dv/df = 0$ (equation (1)), since $\text{R}/p = 1$ when $v = 0$, and substitution in (8) gives $\text{B} = \text{D}[\phi(f_0) - \text{A}f_0 + \frac{1}{2}cf_0^2]/\mu\beta$, so that the only constant left arbitrary is A.

A particular solution of (8) is $v = \text{D}(\kappa - f)/\mu\beta$, where κ is given by

$$\frac{\text{D}}{2\lambda\beta}\cdot\frac{\text{D}^2}{\mu^2\beta^2}\cdot\kappa^2 - \frac{\text{D}}{\mu\beta}\cdot\kappa\text{A} - \text{B} = 0,$$

which leads to

$$\kappa = \frac{A \pm \sqrt{(A - cf_0)^2 + 2c\phi(f_0)}}{c}. \qquad (9)$$

The question of sign will be settled later.

The integral of (8) is found in the usual way by writing $v = D(\kappa - f)/\mu\beta + 1/w$, where w is the new dependent variable, and the equation for w is, after reduction,

$$\frac{dw}{df} = \frac{D}{2\lambda\beta}\frac{1}{\phi(f)} - \frac{A - c\kappa}{\phi(f)} w.$$

The initial conditions are $f = f_0$, $v = 0$, *i.e.*, $1/w = -D(\kappa - f_0)/\mu\beta$, and the solution consistent with these is

$$w = \frac{\mu\beta}{D(f_0 - \kappa)}\left\{\frac{(1-f)(1+\theta f_0)}{(1-f_0)(1+\theta f)}\right\}^{\frac{c'}{1+\theta}} - \frac{D}{2\lambda\beta}\left(\frac{1-f}{1+\theta f}\right)^{\frac{c'}{1+\theta}}\int_f^{f_0}(1+\theta f)^{\frac{c'}{1+\theta}-1}(1-f)^{-\frac{c'}{1+\theta}-1}df, \qquad (10)$$

where

$$c' = A - c\kappa = \pm\sqrt{(A - cf_0)^2 + 2c\phi(f_0)}. \qquad (11)$$

The integral in (10) is easily found, and we have

$$w = \frac{\mu\beta}{D}\left[\left(\frac{1}{f_0 - \kappa} - \frac{c}{2c'}\right)\left\{\frac{(1-f)(1+\theta f_0)}{(1-f_0)(1+\theta f)}\right\}^{\frac{c'}{1+\theta}} + \frac{c}{2c'}\right], \qquad (12)$$

so that the velocity v is given by

$$v = \frac{D}{\mu\beta}\left[\kappa - f + \frac{\dfrac{2c'}{c}}{1 + \left(\dfrac{2c'/c}{f_0 - \kappa} - 1\right)\left\{\dfrac{(1-f)(1+\theta f_0)}{(1-f_0)(1+\theta f)}\right\}^{\frac{c'}{1+\theta}}}\right]. \qquad (13)$$

The question of sign in (11) remains to be settled, but, writing $c'/c = \pm\gamma$, where γ is positive, we find $\kappa = A/c \mp \gamma$, and

$$\frac{2c'/c}{f_0 - \kappa} - 1 = \pm\frac{2\gamma}{f_0 - A/c \pm \gamma} - 1 = \frac{\pm\gamma - f_0 + A/c}{\pm\gamma + f_0 - A/c}.$$

The two expressions for v, given by (13) with the alternatives for c' from (11), are then easily seen to be identical, and we have

$$v = \frac{D}{\mu\beta}\left[\frac{A}{c} - f - \gamma + \frac{2\gamma}{1 + \kappa'\left(\dfrac{1-f}{1+\theta f}\right)^{\frac{\gamma c}{1+\theta}}}\right], \qquad (14)$$

with

$$\kappa' = \frac{\gamma - f_0 + A/c}{\gamma + f_0 - A/c}\cdot\left(\frac{1+\theta f_0}{1-f_0}\right)^{\frac{\gamma c}{1+\theta}}.$$

Equation (1), using (14), gives the pressure p directly in terms of f, whence the maximum pressure can be deduced theoretically, although the equation determining the value of f at maximum is, clearly, only solvable by trial.

The shot travel in terms of f is obtained by integration from (2) and so the problem can be considered as solved.

For the unresisted motion, as is well known, we have $v = D(1 - f)/\mu\beta$, so that $A = c$, $f_0 = 0$ and, therefore, $\gamma = 0$.

The resistance law given by (7) appears to be the only one which will lead to an integral solution in finite terms.

It can be given a variety of expressions by alteration of the value of the constant A, except that, for resistance diminishing as the motion proceeds, A must be greater than cf_0.

The analysis given above is useful so far as it permits a study of the ballistic effects of resistance to the motion, such as the effect on maximum pressure, the position of the shot when the propellant is completely burnt, etc.

3. *The Case of Constant Resistance.*

This case is of importance because it includes the study of the motion of a heavy shot propelled vertically upwards, like the piston of a cylinder, by the combustion of a charge of cordite beneath it.

If the acceleration in such a motion could be determined experimentally the effective pressure on the shot could be calculated absolutely as a mass-acceleration, and thus some light might be thrown on the true pressure-density relation for expanded pressures (*vide* I.B., pp. 349, 350). Friction, of course, should be reduced as much as possible, or could be fairly accurately allowed for at low velocities.

Such experiments may be regarded as the logical extension of close-vessel investigations.

When R is constant equation (3) reduces to

$$\phi(f)\frac{d^2v}{df^2} + \{av + \phi'(f)\}\left(\frac{dv}{df} + b\right) = 0, \quad \ldots \ldots \quad (15)$$

where $a = D/\lambda\beta$, $b = D/\mu\beta$, so that $ab = c$ (*vide* equations (7)).

The absence of R in (15) is to be expected as, when constant, it can be eliminated from (1) by a mere change of units.

By writing $w = av + \phi'(f)$, (15) becomes

$$\phi(f)\frac{d^2w}{df^2} + w\left(\frac{dw}{df} + c'\right) = 0, \quad \ldots \ldots \quad (16)$$

where $c' = c + 2\theta$, taking $\phi(f) = (1 - f)(1 + \theta f)$.

At the start of the motion $v = 0$, $dv/df = 0$ as before, and $f = f_0$, say, $(f_0 < 1)$, where f_0 is calculated from the amount of propellant burnt to make the pressure equal to the resistance R.

Thus, for the equation (16) the initial conditions are

$$f = f_0, \qquad w = \phi'(f_0), \qquad dw/df = \phi''(f_0).$$

The range of f is from f_0, where $0 < f_0 < 1$, to $f = 0$, the value at burnt.

Since $f \neq 1$ the singularities due to the term $1 - f$ in $\phi(f)$ are avoided, so that there is an *a priori* possibility of constructing a solution valid for the range required.

The equation (16) has no first integral of the type (4), unless, as can easily be shown, $c' = 2\theta$, *i.e.*, $c = 0$, which has no particular ballistic importance, so that there appears to be no possibility of an explicit integral being obtained as previously.

We consider, to begin with, the equation (16) for the constant-burning-surface shape, *i.e.*, with $\phi(f) = 1 - f$ and with $c' = c$, since $\theta = 0$.

It will be convenient to take $\phi(f) \equiv 1 - f$, *i.e.*, the fraction of propellant burnt, as the independent variable and we shall write it as ξ.

Then, using suffixes to denote differential coefficients with respect to ξ, equation (16) becomes

$$\xi w_2 + w(c - w_1) = 0. \qquad (17)$$

Now, in this case, $\phi'(f_0) = -1$, $\phi''(f_0) = 0$, and so the initial conditions are $f = f_0$, *i.e.*, $\xi = 1 - f_0 = \xi_0$, say, $w = -1$, $w_1 = 0$, and ξ_0 is a small quantity, since usually f_0 is only slightly less than unity.

A formal solution in series consonant with the initial conditions, is therefore

$$w + 1 = \frac{(\xi - \xi_0)^2}{2\,!}(w_2)_0 + \frac{(\xi - \xi_0)^3}{3\,!}(w_3)_0 + \ldots, \qquad (18)$$

where $(w_2)_0$, $(w_3)_0$, etc., denote the values of w_2, w_3, etc., for $\xi = \xi_0$.

These initial values of the differential coefficients can be calculated in succession from (17), and we have

$$(w_2)_0 = c/\xi_0, \qquad (w_3)_0 = -2c/\xi_0^2, \qquad (w_4)_0 = 3\,!\ c/\xi_0^3 - c^2/\xi_0^2,$$

and, in general, we may write

$$(w_n)_0 = (-1)^n \frac{(n-1)\,!\ c}{\xi_0^{\,n-1}} + \frac{{}_nA_2 c^2}{\xi_0^{\,n-2}} + \frac{{}_nA_3 c^3}{\xi_0^{\,n-3}} + \ldots \qquad (19)$$

where the coefficients ${}_nA_2$, ${}_nA_3$, ... are functions of n, and the last term of the series (19) is ${}_nA_{\frac{n}{2}} c^{\frac{n}{2}} \Big/ \xi^{\frac{n}{2}}$ if n be even, and ${}_nA_{\frac{n-1}{2}} c^{\frac{n-1}{2}} \Big/ \xi_0^{\frac{n+1}{2}}$ if n be odd.

Putting $\eta = (\xi - \xi_0)/\xi_0$, we have

$$\frac{(\xi - \xi_0)^n}{n\,!}(w_n)_0 = \frac{\eta^n}{n\,!}\,\xi_0{}^n(w_n)_0 = \frac{\eta^n}{n\,!}[(-1)^n(n-1)\,!\;c\xi_0 + {}_nA_2c^2\xi_0{}^2 + {}_nA_3\,c^3\xi_0{}^3 + \ldots],$$

so that the solution (18) can be expressed in the form

$$w + 1 = c\xi_0\left(\frac{\eta^2}{2} - \frac{\eta^3}{3} + \frac{\eta^4}{4}\ldots\right) + c^2\xi_0{}^2\left(\frac{{}_4A_2}{4\,!}\eta^4 + \frac{{}_5A_2}{5\,!}\eta^5 + \ldots\right)$$

$$+ c^3\,\xi_0{}^3\left(\frac{{}_6A_3}{6\,!}\eta^6 + \frac{{}_7A_3}{7\,!}\eta^7 + \ldots\right) + \ldots \quad . \quad . \quad . \quad (20)$$

The coefficient of $c\xi_0$ is $\eta - \log(1+\eta)$, provided $|\eta| < 1$, and the coefficients of $c^2\xi_0{}^2$ and $c^3\xi^3{}_0$ can also be found without much difficulty, but the result of importance is the form of the solution (20).

It shows that it is possible to assume the form

$$w + 1 = c\xi_0\,.\,F(\eta) + c^2\xi_0{}^2\,.\,G(\eta) + c^3\xi_0{}^3\,.\,H(\eta) + \ldots \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$F(\eta)$, $G(\eta)$, $H(\eta)$, etc., being functions of η to be determined.

With η as the independent variable, equation (17) becomes

$$(1+\eta)\frac{d^2w}{d\eta^2} + w\left(k - \frac{dw}{d\eta}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where $k = c\xi_0$, and so is usually a small quantity, and the initial conditions are $\eta = 0$, $w = -1$, $dw/d\eta = 0$.

These are satisfied if, for $\eta = 0$, $F(\eta) = G(\eta) = H(\eta) = \ldots = 0$ and

$$dF(\eta)/d\eta = dG(\eta)/d\eta = dH(\eta)/d\eta = \ldots = 0.$$

Substituting from (21) in (22) and equating the coefficients of k, k^2, k^3, etc., to zero, we have the following series of equations :—

$$\left.\begin{aligned} (1+\eta)F_2 - (1-F_1) = 0 \\ (1+\eta)\,G_2 + G_1 + F(1-F_1) = 0 \\ (1+\eta)\,H_2 + H_1 - FG_1 + G(1-F_1) = 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

etc., etc., where F_1, F_2, etc., stand for $\dfrac{d}{d\eta}F(\eta)$, $\dfrac{d^2}{d\eta^2}F(\eta)$, etc.

Integration of equations (23) gives

$$\left.\begin{aligned} F &= \eta - \log(1+\eta) \\ G &= -\eta + \log(1+\eta) + \tfrac{1}{2}\{\log(1+\eta)\}^2 + \tfrac{1}{6}\{\log(1+\eta)\}^3 \\ H &= \frac{3\eta}{2} - \frac{\eta^2}{4} - \frac{3}{2}\log(1+\eta) + \frac{\eta-1}{2}\{\log(1+\eta)\}^2 - \tfrac{1}{2}\{\log(1+\eta)\}^3 \\ &\qquad - \tfrac{1}{6}\{\log(1+\eta)\}^4 - \tfrac{1}{30}\{\log(1+\eta)\}^5 \end{aligned}\right\} . \quad (24)$$

and clearly these solutions are in agreement with the series in (20).

But the series in (20) would have a sum for only a limited range of η; thus the series coefficient of $c\xi_0$ can be considered as equivalent to $\eta - \log(1+\eta)$ only if $\eta \leq 1$, *i.e.*, if $\xi \leq 2\xi_0$, whereas the function $F(\eta)$ exists for all values of η.

It is not difficult, of course, to apply the process of analytic continuation to the series in (20).

Thus

$$\frac{\eta^2}{2} - \frac{\eta^3}{3} + \frac{\eta^4}{4} - \ldots \qquad \equiv 1 - \log 2 + \eta - 1 - \log\left(1 + \frac{\eta - 1}{2}\right)$$

in the sense that the coefficients of η^2, η^3, etc., in the expansion of the right-hand side are still $\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{4}$, ..., so that the differential equation is satisfied. But this expansion is now legitimate for values of $\eta \leq 3$, and further re-arrangements can be made to extend the range of values of η within which expansion in series is possible.

There is thus a justification for the method of solution of the differential equation expressed by (21) and (24).

The above treatment can be applied equally well to any shape of propellant, except that the determination of the functions F, G, H, etc., becomes more difficult.

For the purposes of ballistic calculation the value of η ranges from zero to $(1 - \xi_0)/\xi_0$,* the value at the end of burning of the propellant, and the number of functions F, G, H, etc., required, will depend on this latter value (and that of c).

It is easily seen that $F(\eta)$ is always positive and increases with η, that $G(\eta)$ is always negative but increases in absolute value as η increases, although remaining always less than $F(\eta)$.

The behaviour of the function $H(\eta)$ is not so obvious, but calculation shows that it is negative and increases in absolute value with η.

Any given numerical case must be treated on its merits, and actual calculation alone can show the number of terms of the series (21) required.

A first approximation to the solution for small resistance is perhaps interesting.

* In the case considered, $(1 - \xi_0)/\xi_0 = f_0/(1 - f_0)$, so that, if $f_0 = 0 \cdot 9$, for example, the values of η to be considered range from 0 to 9.

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Taking only the first term of (21) we have, replacing η in terms of ξ,

$$w + 1 = c\xi_0 \left(\frac{\xi - \xi_0}{\xi_0} - \log\frac{\xi}{\xi_0}\right) \quad . \quad . \quad . \quad (24\text{A})$$

or, translating back into terms of v and f,

$$v = b\left[f_0 - f - (1 - f_0)\log\frac{1-f}{1-f_0}\right]. \quad . \quad . \quad . \quad (25)$$

As $f_0 \to 1$ we get $v = b(1 - f)$, the usual formula for non-resisted motion. To this approximation we find also (details omitted)

$$\log\frac{x}{l} = c(f_0 - f) - \tfrac{1}{2}c(1 - f_0)\left(\log\frac{1-f}{1-f_0}\right)^2,$$

and the equation for determining f_m, the value of f at maximum pressure

$$f_m = f_0 - \frac{1}{c} - (1 - f_0)\log\frac{1-f_m}{1-f_0}.$$

Thus the shot travel to the time of completion of burning of the propellant is given by

$$\log\frac{x}{l} = cf_0 - \tfrac{1}{2}c(1 - f_0)[\log(1 - f_0)]^2,$$

corresponding to $\log x/l = c$ for the case of no resistance, showing that, with resistance, the propellant charge is burnt at less travel.

The equation for f_m also shows that $f_m < f_0 - 1/c$, *i.e.*, $< 1 - 1/c$ the ordinary value for non-resisted motion.

Hence the maximum pressure is increased by resistance, as is, of course, to be expected.

There is one point which should be mentioned in connection with these approximations.

Equation (1) expressed in terms of w and ξ is

$$p = \frac{\mathrm{R}c}{c - \dfrac{dw}{d\xi}}, \quad . \quad . \quad . \quad (26)$$

which gives, with the approximation (24A),

$$p = \mathrm{R}\xi/\xi_0.$$

Now we have $p_0 = \mathrm{R} = \lambda(1 - f_0)/l = \lambda\xi_0/l$, and $\mathrm{R}\xi/\xi_0$ obviously increases to a maximum R/ξ_0, since the greatest value of ξ is unity, the value at burnt.

Hence this gives a maximum pressure equal to λ/l, which is the closed-vessel pressure, and the result cannot be correct.

The reason is that the approximation is weakened by differentiating w as required in (26). The process starting from (25) involves an integration, since

$$v = \frac{dx}{dt} = -\beta p \frac{dx}{df} = -\frac{\lambda\beta(1-f)}{x}\frac{dx}{df}.$$

If the second term $G(\eta)$ in (21) is included the value of the maximum pressure is found to be

$$p_m = \frac{\lambda}{lc} \cdot \frac{e^{\sqrt{\frac{2}{c\xi_0}}}}{e^{\sqrt{\frac{2}{c\xi_0}}} - 1 - \sqrt{\frac{2}{c\xi_0}}} \quad . \; . \; . \quad (e = 2 \cdot 718 \ldots),$$

which, for very small values of ξ_0, reduces to the usual value $p_m = \lambda/lc$. If the resistance to motion is not small the problem apparently can only be attacked by direct numerical integration of the differential equations involved.

4. *Further Consideration of the Equations Pertaining to the Pressure-index Burning Law.*

In the previous paper (*vide* I.B., p. 354, equation (22)), the fundamental equation was obtained in the form

$$y \frac{d^2y}{dF^2} = k\psi(F),$$

where $y = x^{1-\alpha}$, and $\psi(F) = \phi(f)$ where $F = \int df/(\phi(f))^{\alpha}$.

For a constant-burning-surface propellant the equation can be put into the simpler form

$$y \frac{d^2y}{dx^2} = x^m, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (27)$$

where $m = 1/(1-\alpha)$ (I.B., p. 356, equation (33)).

It will now be shown that this equation can be integrated by quadratures for certain particular values of m the restriction $\alpha < 1$, *i.e.*, $m > 1$, being abandoned.

Equation (27) is homogeneous of order n where $n = (m+2)/2$ so that, writing $x = e^{\theta}$, $y = ze^{\frac{m+2}{2}\theta}$,* it becomes

$$\frac{d^2z}{d\theta^2} + (m+1)\frac{dz}{d\theta} + \frac{m(m+2)}{4}z = \frac{1}{z}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

which can be reduced to an equation of the first order.

* Small letters x, y, z are written here for convenience, and x and z no longer have their original meaning of shot travel $+ l$, and fraction of charge burnt respectively.

Equation (28) can be integrated if $m = -1$ or $m = -2$ as well as if $m = \pm 0$ (corresponding to $\alpha = \mp \infty$).

We shall leave aside for the moment the question of initial conditions.

(*a*) $m = -1$, *i.e.*, $\alpha = 2$. The equation becomes

$$\frac{d^2 z}{d\theta^2} - \frac{z}{4} = \frac{1}{z}, \quad \ldots \ldots \ldots \ldots \quad (29)$$

where $x = e^\theta$, $y = ze^{\frac{\theta}{2}}$ and the primitive can be expressed as

$$\pm \theta = \pm \log x = \int^z \frac{dz}{\left(\frac{z^2}{4} + 2 \log z + A\right)^{\frac{1}{2}}} + B, \quad \ldots \ldots \quad (30)$$

A, B being arbitrary constants.

(*b*) $m = -2$, *i.e.*, $\alpha = 3/2$. The equation is

$$\frac{d^2 z}{d\theta^2} - \frac{dz}{d\theta} = \frac{1}{z}, \quad \ldots \ldots \ldots \ldots \quad (31)$$

with $x = e^\theta$, $y = z$, so that only a simple change of independent variable has been made.

A further transformation $z = \zeta e^\theta$ changes (31) to

$$\frac{d^2\zeta}{d\theta^2} + \frac{d\zeta}{d\theta} = \frac{e^{-2\theta}}{\zeta},$$

and, finally writing $\theta = f(\xi)$, brings the equation to the form

$$\frac{d^2\zeta}{d\xi^2} + \left[f'(\xi) - \frac{f''(\xi)}{f'(\xi)}\right]\frac{d\zeta}{d\xi} = \frac{[f'(\xi)]^2 e^{-2f(\xi)}}{\zeta}.$$

Choosing $f(\xi)$ so that the coefficient of $d\zeta/d\xi$ vanishes, we have $f'(\xi) = e^{f(\xi)}$, and the final form is

$$\frac{d^2\zeta}{d\xi^2} = \frac{1}{\zeta}. \quad \ldots \ldots \ldots \ldots \quad (32)$$

This integrates, giving

$$\pm \xi = \int^\zeta \frac{d\zeta}{\sqrt{2 \log a\, \zeta}} + \text{constant},$$

a being an arbitrary constant.

Writing $a\zeta = e^{\chi^2}$ this becomes

$$\pm \xi = \frac{\sqrt{2}}{a}\int^\chi e^{\chi^2} d\chi + \text{constant} = \frac{\sqrt{2}}{a} F(\chi) + \text{constant, say.} \quad \ldots \quad (33)$$

Now $\zeta = ze^{-\theta} = y/x$, so that $\chi^2 = \log(ay/x)$. Also $e^{-f(\xi)} = -\xi +$ constant, *i.e.*, $\theta = f(\xi) = -\log(b - \xi)$, where b is another arbitrary constant. Thus $x = e^\theta = 1/(b-\xi)$ or $\xi = b - 1/x$, and the final form of the solution is, from (33),

$$\pm\left(b - \frac{1}{x}\right) = \frac{\sqrt{2}}{a}\,\mathrm{F}\left(\sqrt{\log\frac{ay}{x}}\right) + \text{constant.} \quad \ldots\ldots \quad (34)$$

The above cases appear to be all that can be dealt with directly by quadratures.

5. *The Reduction of a First Order Equation of the Briot-Bouquet Type.*

We now proceed to transform (27) or rather its equivalent (28) to a first order equation of the Briot-Bouquet type.

The characteristic feature of such equations is the indeterminancy of the differential coefficient at the origin, and it is known that unique regular integrals can be constructed, vanishing with the independent variable, provided certain conditions are fulfilled.*

The initial conditions relating to equation (27) are $x = 0$, $y = y_0$, $dy/dx = 0$ (I.B., p. 357), but $x = 0$ corresponds to $\theta = -\infty$ which is inconvenient.

Also $1/z = e^{\frac{m+2}{2}\theta}/y$, and so, if $\alpha < 1$, *i.e.*, $m > 0$, $1/z$ is zero initially.

Further it is found that

$$\frac{dz}{d\theta} = \frac{z}{y}\left(x\frac{dy}{dx} - \frac{m+2}{2}\,y\right),$$

and therefore $d\theta/dz$ vanishes for $x = 0$, $y = y_0$, $dy/dx = 0$. Hence in the case of $\alpha < 1$ we write $z = 1/\xi$, $d\theta/dz = \eta$ in (28), which becomes

$$\xi^3\frac{d\eta}{d\xi} = -(m+1)\,\xi\eta^2 - \frac{m(m+2)}{4}\,\eta^3 + \xi^2\eta^3, \quad \ldots\ldots \quad (35)$$

with $\xi = 0$, $\eta = 0$, initially.

A further substitution $\eta = v\xi$ reduces this to

$$\xi\frac{dv}{d\xi} = -v - (m+1)\,v^2 + \left[\xi^2 - \frac{m(m+2)}{4}\right]v^3, \quad \ldots\ldots \quad (36)$$

and the solution is required in the neighbourhood of $\xi = 0$. Assuming

$$v = \mathrm{A}_0 + \mathrm{A}_1\xi + \mathrm{A}_2\xi^2 + \ldots,$$

substituting in (36), and comparing coefficients we have, since the constant term must vanish,

$$\mathrm{A}_0 + (m+1)\,\mathrm{A}_0^2 + \frac{m(m+2)}{4}\,\mathrm{A}_0^3 = 0,$$

* An exhaustive treatment of the subject is found in FORSYTH's "Theory of Differential Equations," vol. 2, chap. VI. *Cf.* also INCE, "Ordinary Differential Equations," chap. XIII.

whence

$$A_0 = 0, -2/m \text{ or } -2/(m+2).$$

It is clear that if $A_0 = 0$ all the coefficients vanish so that the solution becomes illusory, and to settle the choice between the remaining values we proceed as follows:

We have

$$\eta = v\xi = A_0\xi + A_2\xi^3 + A_4\xi^5 +, \ldots,$$

since A_1, A_3, A_5, etc., are found to vanish. Hence

$$\frac{d\theta}{dz} = \frac{A_0}{z} + \frac{A_2}{z^3} + \frac{A_4}{z^5} + \ldots,$$

so that

$$\theta = \text{const.} + A_0 \log z - \frac{A_2}{2z^2} - \frac{A_4}{4z^4} - \ldots,$$

i.e.,

$$\log x = \text{const.} + A_0 \log (y/x^{\frac{m+2}{2}}) - \frac{A_2}{2}(x^{\frac{m+2}{2}}/y)^2 - \frac{A_4}{4}(x^{\frac{m+2}{2}}/y)^4 - \ldots$$

or

$$\log [x(y/x^{\frac{m+2}{2}})^{-A_0}] = \text{const.} - \frac{A_2}{2}(x^{\frac{m+2}{2}}/y)^2 - \frac{A_4}{4}(x^{\frac{m+2}{2}}/y)^4 - \ldots \quad . \quad . \quad . \quad (37)$$

Equation (37) must be satisfied by the initial values $x = 0$, $y = y_0$, so that $x(y/x^{\frac{m+2}{2}})^{-A_0}$ must not vanish with x, otherwise the logarithm would involve a negative infinity.

With $A_0 = -2/(m+2)$ we have $x(y/x^{\frac{m+2}{2}})^{-A_0} = y^{\frac{2}{m+2}}$, which avoids the difficulty, so that this is the correct value to take. With $A_0 = -2/(m+2)$, equation (36) gives

$$\left.\begin{aligned} A_2 &= -\frac{4}{(m+1)(m+2)^2} \\ A_4 &= -\frac{4(5m+8)}{(m+1)^2(m+2)^3(2m+3)} \\ A_6 &= -\frac{16(10m^2+34m+29)}{(m+1)^3(m+2)^4(2m+3)(3m+5)} \end{aligned}\right\} , \quad . \quad . \quad . \quad . \quad (38)$$

etc., etc.

In the notation previously used (I.B., p. 358) we put $y = y_0 \mathfrak{L}(z)$, where $z = x^{\frac{m+2}{2}}/y_0^2$, whilst in the ballistic tables (I.B., p. 372) z is replaced by Z, where

$$z = \frac{(2-\alpha)(3-2\alpha)}{(1-\alpha)^2} Z = (m+1)(m+2)Z.$$

With these changes, equation (37) becomes

$$\log \mathfrak{L}(Z) = a_2 \frac{Z}{\mathfrak{L}^2} + a_4 \frac{Z^2}{\mathfrak{L}^4} + a_6 \frac{Z^3}{\mathfrak{L}^6} + \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where, on the right-hand side, ζ is written for $\zeta(Z)$, and

$$a_2 = 1, \quad a_4 = (5m + 8)/2(2m + 3), \quad a_6 = 4(10m^2 + 34m + 29)/3(2m + 3)(3m + 5) \ldots$$

Equation (39) shows the precise form of the solution. Equation (36) is not in the exact Briot-Bouquet form since v does not vanish with ξ, but writing $w = 1 + \frac{m+2}{2} v$, $\chi = \xi^2$, it becomes

$$\chi \frac{dw}{d\chi} = \frac{1}{m+2} w - \frac{2}{(m+2)^2}\chi + \frac{6}{(m+2)^2}\chi w + \frac{(m-2)}{2(m+2)} w^2$$
$$- \frac{6}{(m+2)^2}\chi w^2 - \frac{m}{2(m+2)} w^3 + \frac{2}{(m+2)^2}\chi w^3 \quad . \quad . \quad . \quad (40)$$
$$= \phi(\chi\, w), \text{ say,}$$

and here w vanishes with χ.

This equation is of the Briot-Bouquet type and is known to have a unique regular integral vanishing for $\chi = 0$, which exists over a finite part of the region of existence of the function $\phi(\chi, w)$, *provided* $1/(m+2)$ *is not a positive integer* (*vide* FORSYTH, "Theory of Differential Equations," vol. 2, pp. 143–146).

Since $1/(m+2) = (1-\alpha)/(3-2\alpha)$, the excluded values of α are 2, 5/3, 8/5, 11/7, etc. The solution otherwise is then of the form $w = a_1\chi + a_2\chi^2 + \ldots$, which leads to the solution for v originally found.

6. *The Case of a General Burning Law depending on the Pressure.*

In previous work it was assumed that the rate-of-burning was proportional to a power of the pressure, but, if the function of the pressure which measures the rate of burning remains unspecified, the equation connecting them can be written

$$D\frac{df}{dt} = -\beta\psi(p), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

say, where the form of the equation and the constants D, β are retained to permit of comparison with previous results.

We cannot eliminate p as formerly (*vide* I.B., p. 354), and so an equation connecting p and f must be constructed, which can be done in the following manner.

We have

$$\frac{dx}{dt} = \frac{dx}{df}\cdot\frac{df}{dt} = -\frac{\beta}{D}\psi(p)\frac{dx}{df},$$

so that

$$\frac{d^2x}{dt^2} = \frac{\beta^2}{D^2}\psi(p)\frac{d}{df}\left[\psi(p)\frac{dx}{df}\right] = \frac{p}{\mu},$$

by I.B., equation (13). Also $xp = \lambda\phi(f)$ (I.B., p. 352, equation (11)) whence dx/df can be found and substituted in the above to give an equation connecting p and f.

It is found, after reduction, that introducing the function F (p) given by

$$\psi(p) = p^{3/2}[\mathrm{F}(p)]^{\frac{1}{2}} \quad \ldots \ldots \ldots \ldots \quad (42)$$

the following form is arrived at

$$p\frac{d^2p}{df^2} + \tfrac{1}{2}\left[p\frac{\mathrm{F}'(p)}{\mathrm{F}(p)} - 1\right]\left(\frac{dp}{df}\right)^2 - \tfrac{1}{2}\frac{\phi'(f)}{\phi(f)}\left[p\frac{\mathrm{F}'(p)}{\mathrm{F}(p)} - 1\right]p\frac{dp}{df}$$
$$- \frac{\phi''(f)}{\phi(f)}p^2 + \frac{cp}{\phi(f)\,\mathrm{F}(p)} = 0, \quad . \; . \quad (43)$$

where $c = \mathrm{D}^2/\lambda\mu\beta^2$ as before.

Some general comments will be made upon this equation, and others, later in this paper, but at the moment a special result of some interest will be obtained.

Returning to the previous assumption of rate-of-burning proportional to a power of the pressure, take F $(p) = kp^n$, so that $\psi(p) = k^{\frac{1}{2}}\,p^{\frac{n+3}{2}}$ by (42).

Then equation (43) becomes

$$p\frac{d^2p}{df^2} + \tfrac{1}{2}(n-1)\left(\frac{dp}{df}\right)^2 - \tfrac{1}{2}(n-1)\frac{\phi'(f)}{\phi(f)}p\frac{dp}{df} - \frac{\phi''(f)}{\phi(f)}p^2 + \frac{cp^{1-n}}{k\phi(f)} = 0,$$

which can be written

$$p^{-\frac{n-3}{2}}\frac{d}{df}\left[p^{\frac{n-1}{2}}\frac{dp}{df}\right] - \tfrac{1}{2}(n-1)\frac{\phi'(f)}{\phi(f)}p\frac{dp}{df} - \frac{\phi''(f)}{\phi(f)}p^2 + \frac{cp^{1-n}}{k\phi(f)} = 0,$$

so that, writing q for $p^{\frac{n+1}{2}}$, the equation is

$$\frac{d^2q}{df^2} - \tfrac{1}{2}(n-1)\frac{\phi'(f)}{\phi(f)}\frac{dq}{df} - \tfrac{1}{2}(n+1)\frac{\phi''(f)}{\phi(f)}q + \tfrac{1}{2}(n+1)\frac{c}{k\phi(f)}\cdot\frac{1}{q} = 0.$$

Finally, changing the independent variable to F, where $d\mathrm{F}/df = -[\phi(f)]^{\frac{n-1}{2}}$ we have

$$\frac{d^2q}{df^2} - \tfrac{1}{2}(n+1)\frac{\phi''(f)}{\{\phi(f)\}^n}q + \tfrac{1}{2}(n+1)\frac{c}{k\{\phi(f)\}^n}\cdot\frac{1}{q} = 0. \quad . \; . \; . \; . \; . \quad (44)$$

With $\phi(f) = 1 - f$, we have $\phi''(f) = 0$ and $\mathrm{F} = \dfrac{2}{n+1}(1-f)^{\frac{n+1}{2}} = \dfrac{2}{n+1}\{\phi(f)\}^{\frac{n+1}{2}}$,

so that

$$q\frac{d^2q}{d\mathrm{F}^2} = -\tfrac{1}{2}(n+1)\frac{c}{k}\left(\frac{n+1}{2}\mathrm{F}\right)^{-\frac{2n}{n+1}}. \quad \ldots \ldots \quad (45)$$

We thus return to the familiar form of equation (27).

Now consider the two forms, equations (27) and (45), the latter being written as $q(d^2q/d\mathrm{F}^2) \propto \mathrm{F}^{-\frac{3-2\alpha}{1-\alpha}}$, putting $(n+3)/2 = \alpha$. The ballistic problem may be considered as solved when either of these equations is integrated, numerically or otherwise, since the various ballistic elements may be calculated from the solutions so obtained.

If $\alpha = 5/4$, for example, (45) gives $q\,(d^2q/d\mathrm{F}^2) \propto \mathrm{F}^2$, whilst (27) takes the form $y(d^2y/dx^2) = x^2$ for $\alpha = \frac{1}{2}$. Thus the cases $\alpha = \frac{1}{2}$, $\alpha = 5/4$ are mathematically equivalent.

There are similar correspondences between the cases $\alpha = 0$, $\alpha = 4/3$; $\alpha = 2/3$ $\alpha = 6/5$, etc., so that all the cases for values of α from 0 to 1 are paralleled by cases from $\alpha = 4/3$ to $\alpha = 1$.

It is interesting to notice that for $\alpha = 2$ the cases coincide, while for $\alpha = 3/2$, $\alpha = \pm\infty$ they interchange with each other.

These are the cases integrable by quadratures previously discussed (p. 273).

7. *The Connection between the Equation for Resisted Motion and that expressing the case of the General Burning Law.*

In a previous section the case of resisted motion with rate-of-burning proportional to pressure was discussed, but to permit of comparison with equation (43) above it is necessary to assume the resistance R to be a function, $\chi(p)$, say, of the pressure p.

Proceeding as before from equations (1) and (2) the equation connecting p and f is found to be

$$\frac{d^2p}{df^2} - \frac{1}{p}\left(\frac{dp}{df}\right)^2 + \frac{\phi'(f)}{\phi(f)}\frac{dp}{df} - \frac{\phi''(f)}{\phi(f)}\,p - \frac{c}{\phi(f)}[\chi(p) - p] = 0. \quad \ldots \quad (46)$$

This is similar in type to (43), but to agree with (43) we must have $\mathrm{F}(p) = k/p$, where k is a constant, so that by (42), $\psi(p) \propto p$.

Also we find that $\chi(p) = (k-1)\,p/k$, so that only in the simple case of rate-of-burning and of resistance directly proportional to the pressure are the equations (43) and (46) the same.

If we attempt to reduce (46) to an equation of the RICCATI type as previously, by choice of the function $\chi(p)$, we must inevitably be led back to the solution given in the earlier part of the paper (§ 2), but, though of somewhat academic interest it is true, there is a choice of $\chi(p)$ which permits of integration.

For the equation (46) can be written (using primes to denote differentiations)

$$\frac{d}{df}\left(\frac{p'}{p}\right) + \frac{\phi'}{\phi}\cdot\frac{p'}{p} - \frac{\phi''}{\phi} - \frac{c}{\phi}\cdot\frac{\chi(p) - p}{p} = 0,$$

so that the substitution $p = e^q$ gives

$$q'' + \frac{\phi'}{\phi}\,q' - \frac{\phi''}{\phi} - \frac{c}{\phi}\,\frac{\chi(e^q) - e^q}{e^q} = 0.$$

This equation becomes linear if $\chi(e^q) = e^q + ke^q q$, where k is a constant, *i.e.*,

$$\chi(p) = p(1 + k \log p),$$

and this form for the resistance leads to the equation

$$\phi q'' + \phi' q' - ckq = \phi''. \quad \ldots \ldots \ldots \ldots \quad (47)$$

With $\phi(f) = (1-f)(1+\theta f)$ this equation is

$$(1-f)(1+\theta f) q'' - (1 - \theta + 2\theta f) q' - ckq + 2\theta = 0,$$

and, changing the independent variable to η, say, where $f = 1 - \frac{1+\theta}{\theta}\eta$ it becomes

$$\eta(1-\eta)\frac{d^2q}{d\eta^2} + (1-2\eta)\frac{dq}{d\eta} - \frac{ck}{\theta}q + 2 = 0. \quad \ldots \ldots \quad (48)$$

Omitting the constant term this is the differential equation of the hypergeometric series $F(\alpha, \beta, \gamma, \eta)$ with $\gamma = 1$, $\alpha, \beta = \pm\sqrt{1 - \frac{4ck}{\theta}}$.*

One solution is $F(\alpha, \beta, 1, \eta)$ and since $\gamma = 1$ the second solution contains a logarithmic term and can be found by the FROBENIUS process.†

The solution of (48) can, theoretically, be found by putting $q = F(\alpha, \beta, 1, \eta)\, u$ and solving the resulting linear equation for u, although the integrals involved appear intractable. (N.B.—There is no bother about the convergency of the series, since the maximum value of η is $\theta/(1+\theta)$, $(f = 0)$, which is < 1.)

As usual the case $\theta = 0$ leads to simplifications, for then equation (47) becomes $(1-f)q'' - q' - ckq = 0$. Putting now $ck(1-f) = \eta$ we have

$$\eta\frac{d^2q}{d\eta^2} + \frac{dq}{d\eta} + q = 0,$$

a solution of which is $q = J_0(2\eta^{\frac{1}{2}})$, so that the complete primitive can be expressed in terms of BESSEL's functions of zero order.

8. *A Method for the Solution of the Typical First Order Differential Equation of Ballistics.*

The preceding discussion of various ballistic problems has shown that in many cases the problem can be reduced to the solution of a non-linear differential equation of the first order, the general type of which is the following

$$\frac{dy}{dx} = P_0 + P_1 y + P_2 y^2 + P_3 y^3, \quad \ldots \ldots \ldots \ldots \quad (49)$$

where P_0, P_1, P_2, etc., are functions of x (*cf.* equation (36)).

* FORSYTH, "Differential Equations," chap. VI.

† FORSYTH, *loc. cit.*, pp. 251, 252.

We shall find it possible to consider the more general form

$$\frac{dy}{dx} = P_0 + P_1 y + P_2 y^2 + P_3 y^3 + P_4 y^4 + \ldots, \qquad (50)$$

where the number of terms on the right-hand side is not limited to four, but it is of interest to notice that equation (49) can be reduced to the "canonical" form

$$\frac{dY}{dX} = Y^3 + P(X), \qquad (51)$$

as follows.

The substitution $y = Yf(x) + g(x)$ where $f(x)$, $g(x)$ are functions to be determined. denoting differentiations by primes, and the functions by f, g, gives simply

$$f\frac{dY}{dx} = (-g' + P_0 + P_1 g + P_2 g^2 + P_3 g^3) + Y(-f' + P_1 f + 2P_2 fg + 3P_3 fg^2) + Y^2(P_2 f + 3P_3 f^2 g) + P_3 f^3 Y^3.$$

Choosing $g = -P_2/3P_3$ makes the term in Y^2 vanish, and then the coefficient of Y is zero if $f'/f = P_1 - P_2^2/3P_3$, so that $f(x)$ can be determined by integration.

Finally a change of the independent variable leads to the form (51).*

Returning now to equation (50), we assume a solution in the form

$$\text{Const.} = L_0 + L_1 y + L_2 y^2 + \ldots,$$

where L_0, L_1, L_2, etc., are functions of x.

Then the following identical relation in y must hold

$$0 = (L'_0 + L'_1 y + L'_2 y^2 + \ldots) + (L_1 + 2L_2 y + 3L_3 y^2 + \ldots)(P_0 + P_1 y + P_2 y^2 + \ldots),$$

primes denoting differentiations with respect to x.

Thus, to determine the functions L, we have the following series of linear differential equations

$$\left.\begin{aligned} L'_0 + P_0 L_1 &= 0 \\ L'_1 + 2P_0 L_2 + P_1 L_1 &= 0 \\ L'_2 + 3P_0 L_3 + 2P_1 L_2 + P_2 L_1 &= 0 \\ \cdots\cdots\cdots& \\ L'_{n-1} + nP_0 L_n + (n-1) P_1 L_{n-1} + (n-2) P_2 L_{n-2} + \ldots + P_{n-1} L_1 &= 0 \end{aligned}\right\} \qquad (52)$$

It is clear that L_n can be expressed in terms of L'_0, L''_0, ... $L_0^{(n)}$ in the form

$$L_n = {}_nX_1 L'_0 + {}_nX_2 L''_0 + \ldots + {}_nX_n L_0^{(n)}, \qquad (53)$$

${}_nX_1$, ${}_nX_2$, etc., being functions of x and n.

* This transformation is cited in a paper by M. Kouvensky ('C. R. Acad. Sci.,' Paris, vol. 193, pp. 571, 572 (1931)), who gives a reference to P. Appell ('J. de Liouville,' 4, serie 5, p. 370 (1889)).

Thus ${}_1X_1 = -1/P_0$, ${}_2X_1 = -(P'_0 - P_0P_1)/P_0^3$, ${}_2X_2 = 1/2P_0^2$, etc., etc.

Substituting for L_n from (53) in the last of equations (52) we get

$$[{}_{n-1}X_1L''_0 + {}_{n-1}X_2L'''_0 + \ldots + {}_{n-1}X_{n-1}L_0^{(n)}] + n \,.\, P_0\,[{}_nX_1\,L'_0 + {}_nX_2L''_0 + \ldots + {}_nX_nL_0^{(n)}] + \left[\frac{d}{dx}({}_{n-1}X_1)\,L'_0 + \frac{d}{dx}({}_{n-1}X_2)\,L''_0 + \ldots + \frac{d}{dx}({}_{n-1}X_{n-1})\,L_0^{(n-1)}\right] + (n-1)\,P_1\,[{}_{n-1}X_1L'_0 + {}_{n-1}X_2L''_0 + \ldots + {}_{n-1}X_{n-1}L_0^{(n-1)}] + \ldots + \ldots = 0. \quad (54)$$

Equating to zero the coefficient of $L_0^{(n)}$ gives

$$nP_0 \,.\, {}_nX_n + {}_{n-1}X_{n-1} = 0,$$

whence

$${}_nX_n = \frac{(-1)^n}{n\,!\,P_0^{\,n}}. \quad (55)$$

Similarly, from the coefficient of $L_0^{(n-1)}$, we have

$${}_{n-1}X_{n-2} + \frac{d}{dx}({}_{n-1}X_{n-1}) + n \,.\, P_0\; {}_nX_{n-1} + (n-1)\,P_1\; {}_{n-1}X_{n-1} = 0,$$

and, using (55), this difference equation can be solved to give ultimately

$${}_nX_{n-1} = \frac{(-1)^{n-1}\,(P'_0 - P_0P_1)}{2\,!\,(n-2)\,!\,P_0^{\,n+1}}. \quad (56)$$

Proceeding in this way the functions ${}_nX_{n-2}$, ${}_nX_{n-3}$, etc., can be calculated from difference equations making use of results previously obtained.

It is not necessary to give the details here, but the following further results may be quoted :—

$$\left.\begin{aligned}
{}_nX_{n-2} &= \frac{(-1)^{n-1}}{P_0^{\,n+2}}\left[\frac{B}{3\,!\,(n-3)\,!} - \frac{A^2}{8\,(n-4)\,!}\right]\\
{}_nX_{n-3} &= \frac{(-1)^{n-1}}{P_0^{\,n+3}}\left[\frac{C}{4\,!\,(n-4)\,!} - \frac{AB}{12\,(n-5)\,!} + \frac{A^3}{48\,(n-6)\,!}\right]\\
{}_nX_{n-4} &= \frac{(-1)^{n-1}}{P_0^{\,n+4}}\left[\frac{D}{5\,!\,(n-5)\,!} - \frac{2B^2 + 3AC}{144(n-6)\,!} + \frac{A^2B}{48\,(n-7)\,!} - \frac{A^4}{384\,(n-8)\,!}\right]
\end{aligned}\right\} \quad (57)$$

where

$$A = P'_0 - P_0P_1,\; B = P_0P''_0 - P_0^2P'_1 + 2P_0^3P_2 - 3P'^2_0 + 4P_0P'_0P_1 - 2P_0^2P_1^2,$$

etc., so that the functions A, B, C, D, etc., are calculable in terms of the P functions and their successive derivatives.

Now consider the form of the solution. Using (53), the expression

$$L_0 + L_1y + L_2y^2 + \ldots$$

can be written as

$$L_0 + y\,({}_1X_1D)\,L_0 + y^2\,({}_2X_1D + {}_2X_2D^2)\,L_0 + y^3\,({}_3X_1D + {}_3X_2D^2 + {}_3X_3D^3)L_0 + \ldots,$$

where $D \equiv d/dx$.

This is equivalent to

$$\begin{aligned}&(1 + {}_1X_1yD + {}_2X_2y^2D^2 + \ldots + {}_nX_ny^nD^n + \ldots)L_0\\ &+ y\,({}_2X_1yD + {}_3X_2y^2D^2 + \ldots + {}_{n+1}X_ny^nD^n + \ldots)L_0\\ &+ y^2\,({}_3X_1yD + {}_4X_2y^2D^2 + \ldots + {}_{n+2}X_ny^nD^n + \ldots)L_0\\ &+ \ldots + \ldots\end{aligned}$$

Write $\sum_0^\infty {}_nX_ny^nD^n \equiv S_1$, and take ${}_0X_0$ as equivalent to unity, then, from equation (55), we have

$$S_1 \equiv \sum_0^\infty (-1)^n \frac{y^nD^n}{n!\;P_0{}^n} \equiv e^{-\frac{yD}{P_0}},$$

so that

$$S_1L_0 = (e^{-\frac{yD}{P_0}})\,L_0 = (e^{-\frac{yD}{P_0}})\,f(x) \quad \text{if } L_0 = f(x)$$

$$= f(x - y/P_0).$$

Similarly, if $\sum_2^\infty {}_nX_{n-1}y^{n-1}D^{n-1} \equiv S_2$ then by (56),

$$S_2 \equiv \sum_2^\infty \frac{(-1)^{n-1}Ay^{n-1}D^{n-1}}{2!\;(n-2)!\;P_0{}^{n+1}} = -\frac{A}{2P_0{}^2}\frac{yD}{P_0}\,e^{-\frac{yD}{P_0}}.$$

Hence

$$S_2L_0 = S_2f(x) = -\frac{A}{2P_0{}^2}\frac{yD}{P_0}f\left(x - \frac{y}{P_0}\right) = -\frac{Ay}{2P_0{}^3}f'\left(x - \frac{y}{P_0}\right),$$

since the operator D acts on x alone.

As a further result we quote

$$S_3L_0 = \sum_3^\infty {}_nX_{n-2}y^{n-2}D^{n-2}L_0 = \frac{By}{6P_0{}^5}f'\left(x - \frac{y}{P_0}\right) + \frac{A^2y^2}{8P_0{}^6}f''\left(x - \frac{y}{P_0}\right).$$

The solution of (50) can, therefore, be expressed in the form

$$\left.\begin{aligned}\text{Const.} = f\left(x - \frac{y}{P_0}\right) - \frac{Ay^2}{2P_0{}^3}f'\left(x - \frac{y}{P_0}\right) + \frac{By^3}{6P_0{}^5}f'\left(x - \frac{y}{P_0}\right)\\ + \frac{A^2y^4}{8P_0{}^6}f''\left(x - \frac{y}{P_0}\right) - \frac{Cy^4}{4!\;P_0{}^7}f'\left(x - \frac{y}{P_0}\right)\ldots\end{aligned}\right\}, \quad . \quad . \quad (58)$$

where, so far, $f(x)$ is a perfectly arbitrary function.

Taking $f(x) = x$ the solution (58) becomes

$$\text{Const.} = x - \frac{y}{P_0} - \frac{Ay^2}{2!\,P_0^3} + \frac{By^3}{3!\,P_0^5} - \frac{Cy^4}{4!\,P_0^7} + \frac{Dy^5}{5!\,P_0^9} - \frac{Ey^6}{6!\,P_0^{11}} + \ldots \quad (59)$$

and it is fairly clear that the solution (58) can be obtained from (59) by expanding the functional form $f\left(x - \frac{y}{P_0} - \frac{Ay^2}{2!\,P_0^3} \ldots\right)$ so that the solution (59) is in as general a form as is necessary. The successive functions A, B, C, D, E, etc., are clearly to be determined as follows :—

We have

$${}_1X_1 = -1/P_0, \qquad {}_2X_1 = -A/2!\,P_0^3, \qquad {}_3X_1 = B/3!\,P_0^5 \ldots$$

so that we have to try to find the general function ${}_nX_1$. Equating to zero the coefficient L'_0 in (54) we have

$$\frac{d}{dx}({}_{n-1}X_1) + nP_0\,{}_nX_1 + (n-1)\,P_1\,{}_{n-1}X_1 + (n-2)\,P_2\,{}_{n-2}X_1 + \ldots + \ldots = 0,$$

and, putting $n = 3$, 4, etc., in succession, we get the following series of equations :—

$$\left.\begin{aligned} A &= P'_0 - P_0P_1 \\ B &= 2P_0P_1 \,.\, A + 2P_0^3P_2 + P_0 \,.\, A' - 3P_0' \,.\, A \\ C &= 3P_0P_1 \,.\, B - 3\,.\,2\,P_0^3P_2 \,.\, A - 3\,.\,2\,.\,1\,P_0^5\,P_3 + P_0 \,.\, B' - 5P'_0 \,.\, B \\ D &= 4P_0P_1 \,.\, C - 4\,.\,3\,P_0^3P_2 \,.\, B + 4\,.\,3\,.\,2\,P_0^5P_3 \,.\, A \\ &\qquad + 4\,.\,3\,.\,2\,P_0^7\,P_4 + P_0 \,.\, C' - 7P'_0C \\ &\ldots \end{aligned}\right\} \quad (60)$$

The law of formation of these equations is clear, so that the functions can be calculated in succession.

As an example, taking the form (51), or rather the equation

$$\frac{dy}{dx} = 1 + \gamma y^3, \quad \ldots \quad (61)$$

where γ is a function of x, as the equation in this shape leads to simpler results, we have $P_0 = 1$, $P_1 = P_2 = 0$, $P_3 = \gamma$, $P_4 = P_5 = \ldots = 0$, and the solution of (61) is expressed in the form

$$\text{Const.} = x - y + \frac{\gamma y^4}{4} - \frac{\gamma' y^5}{20} + \frac{\gamma'' y^6}{120} - \frac{\gamma''' + 120\gamma^2}{840}\,y^7 + \frac{y^{iv} + 450\gamma y'}{6720}\,y^8 + \ldots \quad (62)$$

9. *Another Form for the Solution of the Equation $dy/dx = y^3 + P(x)$.*

We may obtain another form for the solution by reversing the series in equation (59). Denoting the constant by a we have as the solution

$$a - x = L_1 y + L_2 y^2 + L_3 y^3 + \ldots,$$

which, on reversing the series, gives

$$y = M_1 z + M_2 z^2 + M_3 z^3 + \ldots, \qquad (63$$

where z is written for $a - x$, and the M's are functions of x. Differentiating (63) with respect to x,

$$\frac{dy}{dx} = (M_1 z + M'_2 z^2 + \ldots\ ) - M_1 - 2M_2 z - 3M_3\, z^2 \ldots$$
$$= P + (M_1 z + M_2 z^2 + \ldots)^3$$

by the differential equation, and comparison of the coefficients of respective powers of z determines the M functions.

It is found that

$$M_1 = -P\,;\ M_2 = -\frac{1}{2\,!}P'\,;\ M_3 = -\frac{1}{3\,!}P''\,;\ M_4 = -\frac{1}{4\,!}P''' + \frac{1}{4}P^3\,;$$
$$M_5 = -\frac{1}{5\,!}P^{iv} + \frac{9}{20}P^2P', \text{ etc.},$$

and in general M_n contains the term $-\dfrac{1}{n\,!}P^{(n)}$.

Now it is easily seen that, since $z = a - x$,

$$-Pz - \frac{P'}{2\,!}z^2 - \frac{P''}{3\,!}z^3 - \frac{P'''}{4\,!}z^4 \ldots = \int_a^x P\,dx,$$

and, therefore, the solution is given as

$$y = \int_a^x P\,dx + \frac{P^3}{4}(x-a)^4 - \tfrac{9}{20}P^2P'(x-a)^5 + (\tfrac{19}{120}P^2P'' + \tfrac{11}{40}PP'^2)(x-a)^6$$
$$+ (-\tfrac{17}{420}P^2P''' - \tfrac{41}{210}PP'P'' - \tfrac{2}{35}P'^3 + \tfrac{3}{28}P^5)(x-a)^7 + \ldots \qquad (64)$$

The methods outlined above have been given as an attempt at some general method of approach to the integration of non-linear equations of the first order, and it is realised that many considerations have been omitted such as, for example, the question of the convergency of the series (59) and (64).

10. *The Case* $P_0 = 0$.

If $P_0 = 0$ so that the right-hand side of the differential equation begins with a term involving y, as in equation (36), the above analysis breaks down.

The same method can be used, however, for if the equation is

$$\frac{dy}{dx} = P_1 y + P_2 y^2 + P_3 y^3 + \ldots \qquad (65)$$

assuming a solution of the form

$$L_0 + L_1 y + L_2 y^2 + L_3 y^3 + \ldots = \text{constant},$$

leads to the equations $L'_0 = 0$, $L'_1 + P_1 L_1 = 0$, $L'_2 + 2L_2P_1 + P_2L_1 = 0$, etc., etc. Thus L_0 is a constant which can be taken as zero without loss of generality, and the functions L_1, L_2, etc., can then be determined in succession.

The form of the solution is then

$$L_1 y + L_2 y^2 + L_3 y^3 + \ldots = \text{constant},$$

and the solution which vanishes with x can only be $y = 0$.

This is known to be the case for an equation of the type $dy/dx = y^m f(x, y)$, where m is a positive integer.*

11. *Some General Considerations.*

The equations discussed in this paper are, with few exceptions, non-linear and of the second order, although, in particular cases, reduction to an equation of the first order can be effected.

In the pure mathematical treatment of such equations it is usual to consider the variables as complex, and the questions which are dealt with concern the existence, or otherwise, of regular integrals, and the classification of the singularities of the solutions.

Such singularities may be poles, branch points or essential singularities, and they may be fixed or movable, *i.e.*, independent or otherwise of the prescribed initial values of the variables.

In applications to practical calculations, such as occur in ballistics, these considerations are usually ignored, and the solutions are sought by numerical methods, the worker trusting to the physical nature of the problem as a guide to correct results.

But numerical methods are applicable only to each particular problem in turn, and the broad survey which, when possible, definite integration of the differential equations would give is lacking.

It is, therefore, not without interest to examine the equations which arise to see if they conform to any of the types which are of importance in the mathematical theory.

It has already been shown (pp. 275–278) that the fundamental ballistic equation for the case of a constant-burning-surface propellant can be transformed into an equation of the Briot-Bouquet type, which is known to possess a regular analytic solution under certain conditions. These conditions are seen to involve the pressure index α, and the regular solutions do not exist for certain values of α; the excluded values, however, do not lie between 0 and 1, so that all the practical ballistic cases are covered.

This result is of importance as it establishes the validity of the numerical integrals given in I.B. (p. 375).

All the other problems treated in this paper lead to irreducible second order equations, as do also the ballistic investigations in I.B. for shapes of propellant other than constant-burning surface. The chief matter of theoretical interest in such cases appears to be the occurrence, or otherwise, in the solutions of movable branch points and essential singularities.

A full and convenient discussion of this problem is given by Ince (*loc. cit.*, chap. XIV), who arrives at a number of typical equations which have all their critical points fixed, *i.e.*, independent of the prescribed initial conditions.

* *Vide* Forsyth, "Theory of Differential Equations," vol. 2, pp. 44, 45, and pp. 63 *et seq.*

It is first shown (INCE, *loc. cit.*, p. 321) to be necessary that the form of the equation should be

$$\frac{d^2y}{dx^2} = \mathrm{P}\left(\frac{dy}{dx}\right)^2 + \mathrm{Q}\frac{dy}{dx} + \mathrm{R},$$

where P, Q, R are rational functions of y with coefficients analytic in x, and the ballistic equations are of this type.

It becomes of interest, then, to examine if these latter conform in any way to the equations discussed by INCE.

As an example take equation (16), and write it in terms of x and y as

$$\phi(x)\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx} + c'\right) = 0,$$

where, in general, $\phi(x) = (1 - x)(1 + \theta x)$, $0 < \theta < 1$ (I.B., p. 350). The transformation $y = 2z\phi(x)$ brings it to the form

$$\frac{d^2z}{dx^2} + 2z\frac{dz}{dx} + \frac{2\phi'(x)}{\phi(x)}\left(\frac{dz}{dx} + z^2\right) + \frac{cz}{\phi(x)} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (66)$$

(since $c' = c + 2\theta$ and $\phi''(x) = -2\theta$) and this is only of the prescribed type if $c = 0$ (INCE, *loc. cit.*, pp. 330, 331).

In this case, writing $\zeta = dz/dx + z^2$, we have $d\zeta/dx + 2\phi'(x)\zeta/\phi(x) = 0$ or $\zeta = k/\{\phi(x)\}^2$, with k an arbitrary constant.

Thus the equation for z is

$$\frac{dz}{dx} + z^2 = \frac{k}{\{\phi(x)\}^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (67)$$

which is of the RICCATI type.

Hence if $c \neq 0$ the integral must involve logarithmic terms (INCE, *loc. cit.*), which confirms the form of the solution already obtained. A special solution of (67) is $z = (a - \theta x)/\phi(x)$ where $a^2 + a(1 - \theta) = k\theta$, so that the complete primitive can be found in the usual way, but $c = 0$ does not correspond to any case of ballistic importance. (N.B.—$c = 0$ if $\mathrm{D} = 0$, *i.e.*, if the propellant size is vanishingly small.)

An equation whose general form appears to agree with that of the canonical equations given by INCE (chap. XIV), is (43), the ballistic equation for the case of a general rate-of-burning law dependent on the pressure. For this equation to have solutions with fixed critical points the coefficient of $(dp/df)^2$ must take one of a number of forms (INCE, p. 326), and, by choice of $\mathrm{F}(p)$, this condition can be secured, but it appears that the terms involving dp/df and p cannot be made to agree with those occurring in the typical equations quoted by INCE.

An examination, admittedly cursory, of (43) (and other equations), therefore, seems to lead to the conclusion that the second-order differential equations of ballistics possess movable critical points. The special quadrature solutions given by (30) and (34) appear to confirm this view.

12. *Summary and Conclusions.*

In this paper an examination is made of various types of differential equation which arise in internal ballistics, when the simple assumptions of rate-of-burning proportional to pressure, or of non-resisted motion are discarded.

The equations are shown to be mostly irreducible, non-linear, and of the second order, but in the case of a propellant shape preserving a constant surface area during burning, they can be reduced to the first order, and finally transformed into equations of the BRIOT-BOUQUET type.

The existence of regular integrals is thus assured, except when the pressure-index α takes values which make $(1-\alpha)/(3-2\alpha)$ a positive integer. An excluded case, for example, is $\alpha = 2$ and for this value the solution can be obtained by quadratures.

It is also shown that the cases for which α varies from 4/3 to 1 depend on the same type of fundamental equation as those for which α is between 0 and 1, though, of course, in different variables; thus these sets of cases are mathematically equivalent.

Most of the first order equations of internal ballistics can be written in the form $dy/dx = P_0 + P_1 y + P_2 y^2 + P_3 y^3$, where P_0, P_1, P_2, etc., are functions of x, and, in the case where no terms beyond y^3 occur, the equation can be transformed into the "canonical" form $dy/dx = y^3 + P\ (x)$.

A method of solution of equations of the above type is suggested as a power series in y, the coefficients being functions of x, which are determined in terms of the function P and its successive derivatives. A reversal of this series leads to the explicit expression of y in terms of x.

The second order equations which arise when the propellant shape is other than of constant burning surface are usually found to be irreducible, and are much less easy to discuss.

The important equations of this order from the pure mathematical point of view are those having fixed critical points, but the differential equations of ballistics, although similar in some respects to such equations, do not appear to possess this property.

Several problems are left undiscussed; thus there is no investigation of the equations arising from Scheme II (the CHARBONNIER equations, I.B., p. 353) or of the problem of resisted motion with a pressure-index burning law, as the analytical complexities appear to be too serious.

The writer is conscious of the many gaps and loose ends in this paper, but he has tried to show something of the analytical difficulties met with in ballistic problems, and the attempts that have, so far, been made to overcome them.

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